

Cantor Paradoxes, Possible Worlds and Set Theory

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Abstract: In this paper, we illustrate the paradox concerning maximally consistent sets of propositions, which is contrary to set theory. It has been shown that Cantor paradoxes do not offer particular advantages for any modal theories. The paradox is therefore not a specific difficulty for modal concepts, and it also neither grants advantages nor disadvantages for any modal theory. The underlying problem is quite general, and affects anyone who intends to use the notion of “world” in its ontology.

Keywords: actualism; Cantor theorem; paradoxes; possible worlds; sets

1. Introduction

In the last decades of the past century and the first of the current century, it was considered by some philosophers that a good way to understand facts as true or false statements is with reference to a totality of “possible worlds” (e.g., Adams [1], Plantinga [2]). When we say that “*p* is necessary”, it should be understood that *p* is true in all possible worlds, and when we say that “*p* is possible”, then it should be understood that there is at least one possible world in which *p* is true. What are the reasons to think that this is an acceptable way of understanding the truth conditions of modal statements? Lewis [3] argues that these different possible worlds are entities of the same nature as the actual world—that is to say, as mereological sums of all entities related to each other in space and time. Lewis implemented this argument in a detailed and complex way, in order to justify that possible worlds should be understood as he postulates, and not as postulated by an actualist philosopher [4]. It is perfectly possible to accept Lewis’ argument and discard the idiosyncratic way in which Lewis explains possible worlds. The crucial point is that there seems to be a very simple reason to accept that, because we believe that things could be different, there are alternative ways in which things could be. If there are alternative ways in which a specific entity might be, there seems to be no difficulty in thinking that there are alternative ways that all things could be—that is, as all possible worlds.

Understanding what the possible worlds are is therefore to understand what makes modal propositions true or false. It seems obvious that there are many true propositions about what might happen, but if such truths exist (and are known to us), then there must be something by virtue of which they are true. It seems intuitively obvious that what makes a modal proposition true are alternative ways in which things could occur, and the totality of possible worlds makes the modal propositions true or false.

An extreme form of modal realism is defended by David Lewis. This view is usually called “modal possibilism”, and opposes those theories maintaining that the actual world is not the same from the ontological point of view as all the remaining plethora of possible worlds. These other theories are usually called “actualist”, as ontological preference is given to the actual world. The work presented here has to do with a set of difficulties affecting actualistic theories of modality. Indeed, if it

is postulated that, in some sense, the only “real” world is today’s world, then how can all possible worlds be understood? The actualist philosopher must in some way say that “there are” all possible worlds, without denying the ontological preference for the actual world. The handiest way to make this specification is by arguing that all possible worlds are abstract constructions made from elements that are already given in the actual world.

2. Possible Worlds and Set Theory

Adams conceived of possible worlds as “world stories” or “complete stories” about how reality could be constituted. That is, “a set which has as its members one of every pair of mutually contradictory propositions, and which is such that it is possible that all of its members be true together” [1] (p. 204). This is not to suggest a parallel universe, but rather to present a way to model the infinitely detailed multiple perspectives that constitute our everyday experience. However, these stories are all possible worlds. These stories may be considered as sets of sentences or as a set of propositions. These stories could theoretically describe the world fully when they are maximally consistent.

Definition 1. A set of propositions S is **maximally consistent** if for every well-formulated proposition p , $p \in S$ or $\neg p \in S$.

The problem presented here directly affects conceptions of all possible worlds associated with “Complete Stories”, but other forms of actualism are also subject to paradoxes of this style, so the examination made here has a general value. The problem arises from a basic consequence of set theory. Cantor proved that the cardinality of the power set of a given set is greater than the cardinality of the set itself [5]. Generally speaking, the cardinality of the power set of a set A , such that $\text{Card}A = n$, is 2^n (i.e., $\text{Card}A = n \rightarrow \text{Card}P(A) = 2^n$). Cantor was interested in the generalization of this result to transfinite numbers [6–8]. If a cardinality that can be assigned to the set of natural numbers, that is, $\text{Card}N = \aleph_0$, then immediately a power set of the set of all natural numbers can be defined, with cardinality equivalent to $\text{Card}P(N) = 2^{\aleph_0} = \aleph_1$. Nothing prevents the definition of the power set of the power set of the set of all natural numbers; that is, $PP(N)$ and its cardinality can be defined as $\text{Card}PP(N) = 2^{\aleph_1} = \aleph_2$. The iteration of the same procedure can generate a “Cantor paradise” of entities, all of infinite numerosity but not equivalent, because in general $\forall n, n < 2^n$. All this is well known. What is relevant to the question at issue here is $\text{Card}P(A) > \text{Card}A$.

Consider that in actualist conceptions, it is argued that a possible world is a maximally consistent set of propositions, formulated in language. For each subset of all these propositions, there will be a proposition that either forms part of the set, or its denial will be part of the set.

Theorem 1. The maximally consistent set in question has a number of propositions as big as its power set, which contradicts Cantor’s theorem.

Proof.

1. For definition 1 $\forall S (S \text{ is maximally consistent}) \leftrightarrow (\forall q (q \in S \vee \neg q \in S) \wedge (S \text{ is consistent}))$.
2. There is a set of propositions A that is maximally consistent.
3. $\text{Card}P(A) > \text{Card}A$.
4. For each set belonging to the power set $P(A)$ there will be a proposition of that set. $\forall S ((S \in P(A) \rightarrow \exists q (q \text{ is about } S)))$.
5. Let r be a similar proposition. By 1 it follows that: $(r \in A) \vee (\neg r \in A)$.
6. As 5 applies to any element of $P(A)$, there will be as many propositions A as sets belonging to the power set of A , so it follows that: $\text{Card}P(A) \leq \text{Card}A$.

However, 6 is in open contradiction with 3. There is therefore a line of reasoning that leads from premises 1 to 4 to a contradiction. So, one of these premises must be rejected:

- It is not plausible to reject the definition of a maximally consistent set of Proposition 1.
- Nor is it reasonable to reject Cantor's Theorem 3.

Open options are:

- Assumption 2 is that there is a set A of propositions that are maximally consistent.
- There is at least one proposition for each subset of the power set of A that is 4.

As 4 is plausible for such reasons, it seems that 2 is to be rejected. That is, it seems that the idea that there is a maximally consistent set of propositions is inconsistent with set theory and must be rejected.

It will be argued that it is reasonable to consider that really the “world” is not a set. The other “structures” that are often argued to actually deliver possible worlds should not be conceived as sets. That is, set theory can be considered as a very important abstract mathematical theory, but not as a fundamental ontology. \square

3. Seven Cantor Paradoxes

Considerations of Cantor paradoxes are evoked by infinite sets, and there is a whole family of problems affecting actualistic conceptions of possible worlds with profound metaphysical applications. Grim [9–11] points to arguments involving the idea that there is a set of all truths, that there is a set of all necessary truths, that there is a set of all falsehoods, and there is a set of all things known by an omniscient being. Bringsjord [12,13] has raised the same difficulty with all sets of states of affairs or facts. Chihara [14], meanwhile, has developed arguments against the idea that there is a set of all possible states of affairs and against the idea that there is a set of all essences. As a final example, David Kaplan refers to a Cantor paradox affecting the assumption of the existence of a totality of possible worlds. This paradox was formulated in M. Davies [15] (p. 262). The argument is satisfactorily answered by Lewis [4] (pp. 104–108). Let us see different forms of these paradoxes.

C1 *There is no set of all truths.*

Suppose we have a set of all truths, let $A = \{t_1, t_2, \dots, t_n\}$. There is a cardinality of the set of all truths $\text{Card}A = n$. However, it turns out that each subset of A may be assigned a true proposition. For example, if $B \subseteq A$, defined as the set $\{p_{i-1}, p_i, p_{i+1}\}$, then there exists at least one truth $p_i \in B$. This truth has to be part of the set A , because it is the set of all truths. However, the set of all truths then has as many elements as the power set of A , $P(A)$. Thus, $\text{Card}A \geq \text{Card}P(A)$, contrary to Cantor's result. This form of reasoning shows that there is no set of all necessary truths and there is no set of all falsehoods, nor is there a set of all truths which are known by an omniscient being. In the case of necessary truths, the difference is that the set A is restricted. Each subset has assigned another as a trivially true and necessary proposition, because every true proposition belonging to a given subset is a necessary fact. In the case of the set of all falsehoods, set A will consist of all the denials of the set that have entered into the central argument about the truths. A false proposition may be assigned to each subset (e.g., if a subset of A is $\{p_1, p_2\}$, then a falsehood can be made by stating that $p_1 \notin \{p_1, p_2\}$) and the argument works again in the same way. Finally, in the case of all true propositions known by an omniscient being, the argument works because the omniscient being—precisely because it is omniscient—must also know each subset of A , which is a subset of A .

C2 *There is no set of all states of affairs or facts.*

This is a very similar argument to 1. This is because the states of affairs or facts are conceived as those which make a proposition ordinarily true (or false, depending on the case). If there is an argument for the truth, then it seems obvious that there will be an argument for those entities that are correlated with true propositions. Let A be the set of all states of affairs in the world. To each subset of A , we can assign a new state of affairs. For example, if there is a subset

$B = \{S_1, S_2\}$, then there is also a state of affairs S_3 constituted by the fact that S_1 belongs to the set B .

C3 *There is no set of propositions or sentences that is maximally consistent.*

The reasoning is as detailed above. This is the argument that seems to directly attack actualistic conceptions, which work with “full stories.”

C4 *There is no set of all possible states of affairs.*

This is a variation of 2 which is based on the total of all current states of affairs. Let A be the set of all possible states of affairs. To each subset of A we may assign a possible state of affairs. For example, if $B \subseteq A$ consists of $\{S_1, S_2\}$, then there is a possible state of affairs and it is consistent, where $S_1 \in B$. This argument seems to go directly against the way of conceiving all possible worlds as a maximum possible states of affairs [16].

C5 *There is no set of all essences.*

This is an argument that has been directed at Plantinga’s modal conception, in which non-current possible entities are represented by a substance that is not found instantiated [2]. The essence is here a set of properties that are satisfied by one and only one individual in all possible worlds. It must be assumed that there are individual essences that are not only objects but also states of affairs. That is, each state of things, actual or merely possible, is assigned a property that is satisfied by this state of affairs, and nothing else in all possible worlds. We use special symbols called *descriptions*, and they are of the form $|xF(x)$ that we can read: *the x that satisfies the function F* . For example, if we want to specify the individual essence of the state of affairs in which the horse Rocinante is starving, we can define a consistent property to be something that is at the same time Rocinante and starving. That is, if R is the individual essence of horse Rocinante ($|RH(R)$, R fulfills the function H of being a horse) and S is the property of being starved ($|RS(R)$ R fulfills the function S of being starved), then there is the property of $|RH(R) \wedge |RS(R)$. It is trivial then that these individual essences or states of affairs can be defined, if there are individual essences for objects that constitute such states of affairs. Let A be the set of all individual essences. Each subset of A can be assigned an individual essence. In effect, let B be a set $B \subseteq A$ composed of $\{E_1, E_2\}$, where E_1 and E_2 are individual essences. We consider that $E_1 = |RH(R) \wedge |RS(R)$ and $E_2 = |QM(Q) \wedge |QKE(Q)$ (for example, E_1 is the individual essence of the state of affairs for Rocinante to be starving and E_2 is the individual essence of being the man (M) Don Quixote (Q) a knight errant (KE)), then it is trivial that there is an individual essence $E_3 = E_1 \cap E_2 = [|RH(R) \wedge |RS(R)] \wedge [|QM(Q) \wedge |QKE(Q)]$ (i.e., the individual essence of the state of affairs for Rocinante to be starving, and Don Quixote a knight errant). Therefore, there are so many individual essences as subsets of the set of all individual essences. This creates the Cantor paradox immediately.

C6 *There is no set of all entities.*

Simply assume that our ontology is based on a principle of mereology as follows: if there are two different objects x and y , then there is a mereological sum $(x + y)$. This is an intuitive plausible principle. Let A be the set of all entities of the world, such that $A = \{x_1, x_2, \dots, x_n\}$. Each subset of $P(A)$ can be assigned a specific entity, constituted by the mereological sum of all entities that are members of that subset. According to the mereological principle indicated, these entities are also entities of the world and must be elements of the set A . It follows, then, that it appears that A has as many elements as its power set, contrary to what has been established by Cantor’s theorem.

C7 *There is no set of all universals.*

Suppose there was a set of all universals $A = \{U_1, U_2, \dots, U_n\}$. There is a power set of A , $P(A)$ whose cardinality must be greater than the cardinality of A . However, for each subset of A ,

we can define a universal complex set. For example, given $B \subseteq A$ such that $B = \{U_1, U_2\}$, then there is a universal set which is the joint instantiation of U_1 and U_2 . It happens that there are so many universal sets, as subsets of the set A , but all these universals belong to A , contrary to what has been established by Cantor's theorem.

4. Cantor Paradoxes and Possible Worlds

As noted above, one can think of possible worlds as “complete stories” about how reality could be constituted. Do not assume that there is a world apart from the actual world where such stories are true, as there can be a story, infinitely detailed and exhaustive with respect to the actual world. However, these stories are all possible worlds. One can think of these stories as sets of sentences in a given language or simply as a set of propositions. The way in which these stories come to fully describe how the world could be made is because they are maximally consistent. The actual world is not the same, from the ontological point of view, as all the remaining plethora of possible worlds. These theories are usually called “actualist”, as ontological preference is given to the actual world. How do these actualist theories affect these various Cantor paradoxes? The actualistic theories are usually classified into four groups [17] (pp. 169–180):

1. Linguistic (or propositional) theory.
2. Combinatorial theory.
3. Plantinga's theory.
4. Theory of possible worlds as properties.

It will be seen here that there is no actualist theory showing advantages with regard to problems of cardinality.

4.1. Linguistics (Propositional) Theories

It is characteristic of these modal concepts that possible worlds are conceived as a “complete story” of how reality could be made. It is not about how such stories are true, but about the stories themselves. If there is a complete specification of how the world could be given, down to the smallest detail for each instant of time, then all that is wanted is for it to be the content of the world; if things were different from how they are currently, it will be reflected in some peculiarity of the “complete story”.

The complete story can be built using a particular language or a set of propositions. In principle, the use of propositions may be more appropriate since it is reasonable to expect that the propositions do not have the expressive limitations that may affect our natural and artificial languages. One may wonder about the constructions of maximally consistent sets of propositions which are included among the linguistic theories. This group is motivated by the fact that the “complete stories” consist of predicates and “complete stories” formed by propositions that have a similar structure. If we appeal to a natural language for the construction of “complete stories”, it is obvious that there will be important inadequacies, because in certain languages like English or Spanish, we have no names for each existing entity (much less for each nonexistent entity that is currently possible). As noted above, the way that a “complete history” typically gets specified is by taking every sentence or formed proposition of the language in question and adding either the sentence (or proposition) or their negation to a consistent set. This procedure generates a whole of maximally consistent sets of sentences or propositions, which are entities that in this instance meet the role of possible worlds.

Definition 2. A state of things is current if the proposition expressing the giving of such a state of things is true.

Definition 3. The current world is determined by the maximally consistent set of sentences and propositions, in which all elements are true.

Definition 4. *A state of things is possible if the proposition expressing this state of things belongs to at least one maximally consistent set.*

Definition 5. *A state of affairs is necessary if the proposition expressing the given of this state of affairs belongs to all maximally consistent sets.*

This way of conceiving all possible worlds is simple and elegant, and perhaps for that reason has proved attractive to many philosophers who have worked in the purification of one or another form of linguistic theory [18–20]. However, theories of this kind are directly affected by the Cantor paradoxes (which can be adapted to maximally consistent sets of sentences or to maximally consistent sets of propositions) of C3 form.

It seems obvious that for each sub-set of the maximally consistent set describing how the world is constituted, there exists a statement or proposition indicating its truth or potential truth if this world is actual. The linguistic theories are therefore directly affected by the Cantor paradoxes.

4.2. Combinatorial Theories

Defenders of combinatorial modal conceptions include Armstrong [21] and Cresswell [22]. In the combinatorial theories, possible worlds are conceived as constructions from a given set of elements that are already given in the current world. These elements are basically objects and properties. In the current world, objects and properties are states of affairs or facts which make true the propositions stating the given states of affairs. The general idea is that these same objects and properties could be combined in other ways; that is, the same objects that already exist in the current world could instantiate other properties and relationships, therefore setting other states of different things existing in the current world. Naturally, there are philosophers who are inclined to replace objects and property by tropes, specifying the states of things differently, but the central idea is the same.

Let $\{x_1, x_2, \dots, x_n\}$ be the set of all objects and $\{P_1, P_2, \dots, P_n\}$ be the set of all properties.

Definition 6. *The set of all possible states of affairs is defined as the set of all ordered pairs (or n -tuples ordinate) of objects and properties $\{ \langle P_1, x_1 \rangle, \langle P_1, x_2 \rangle, \dots, \langle P_1, x_n \rangle, \langle P_2, x_1 \rangle, \langle P_2, x_2 \rangle, \dots, \langle P_2, x_n \rangle, \dots, \langle P_n, x_1 \rangle, \langle P_n, x_2 \rangle, \dots, \langle P_n, x_n \rangle \}$.*

Definition 7. *A given set of all states, independent of other things, may be defined as all possible worlds that are the totality of all possible combinations of states of affairs.*

Using an elementary example, it can be seen how these possible worlds are specified: let S_1 and S_2 be two states of affairs, then there are four possible worlds, namely $\{S_1, S_2\}$, $\{S_1, \neg S_2\}$, $\{\neg S_1, S_2\}$, $\{\neg S_1, \neg S_2\}$.

In the combinatorial design, sets of objects and properties are taken from the objects and properties that are currently in existence. These combinatorial concepts are directly affected by Cantor arguments of the C4 form because they must apply to the set of all possible states of affairs, in which states of affairs each sub-set of this group should be included.

4.3. Plantinga's Theory

Plantinga [2,16] defines possible worlds as maximum possible states of affairs. He does not give clear indications of what constitutes “states of affairs”, but it must be assumed that his conception cannot differ too much from the usual, where the states of affairs:

1. Are given by the characteristics of objects and properties.
2. Are the entities that are true (or false) in relation to sentences or propositions.

A state of “possible” things must be a state of affairs that even if it is not actual, it could be. Moreover, it is said that a state of affairs S is “maximum” if and only if for every state of affairs S^* , S includes S^* or S excludes S^* .

- a. S includes S^* iff, necessarily, if S is actual then S^* is actual.
- b. S excludes S^* iff S and S^* cannot be together.

A maximum state of affairs as understood by Plantinga is a state of affairs which must include any determination or fact which could be an alternative way in which the world is constituted, and Plantinga identifies a possible world in this way. Plantinga adds to the notion of possible states of affairs and maximizes the notion of the “book” world. Given a possible world, there will be a set of true propositions about what happens in that possible world. This set of true propositions about such a world is what is called here the “book” of that world. Plantinga’s conception is affected by the Cantor paradox in several ways.

1. First, Plantinga seems to require a set of all possible states of affairs (i.e., something that seems to be quantifying its definition of a “state of maximum things”, and the definitions of “inclusion” and “exclusion”), and this is what makes his conception sensitive to the argument C4.
2. Second, even if Plantinga does not conceive of possible worlds as sets of propositions, he contends that there is a set of propositions associated with each possible world which must be maximally consistent. “Books” are susceptible to falling into the Cantor paradox of the C3 form, just as the modal linguistic conceptions.
3. Third, Plantinga’s design disclaims possible objects using a set of individual essences. The postulation of a set of individual essences, however, seems to be affected by Cantor paradoxes of the C4 form.

4.4. Theory of Possible Worlds as Properties

In this modal conception, possible worlds are conceived as universal or structural properties that specify how the world would be made if things were different [23,24].

Definition 8. A *structural universal* is a universal, that is, a certain determination that it is by its nature apt to be predicated by many things that arise from the complex of other more basic universals.

Example 1. The universal “water molecule” is instantiated for something in which there are three parts, one of which is the instance of the universal “oxygen atom”, two of which instantiate the universal “hydrogen atom”, and these three parts are related together instantiating other relational universals.

The general idea is that a possible world (a possible way in which the world could be constituted) would be given by a highly complex structural universal in which every given detail would be found, and every decision that that world could have. In this conception, the difference between the present world and the other possible worlds is that the present world is the only one that is instantiated. The other possible worlds that are universal are not instantiated (but could be).

Definition 9. A *state of affairs* S is *possible* if there is at least one maximum structural universal U , such that if U was instantiated, then S would be given.

Definition 10. A *state of affairs* S is *necessary* if for all maximum structural universals U , if U is instantiated, then S would be given.

It is crucial for the identification of a possible world with a structural universal that the universal in question is, in any sense of the word, “maximum”. In the case of modal theory based on universals, the idea is that a possible world is a “maximum” structural universal because it comprehensively

describes the possible world provided. A universal in question is able to encode how the world would be, because:

1. It is a specification of a “maximum” individual—that is, an individual such that every individual is part of it.

Definition 11. *Formally, we can define a maximum individual as*

$$\forall x [(x \text{ is maximum}) \leftrightarrow \forall y (y \text{ is part of } x)].$$

In mereology every object is part of itself, even if it is not the only part of itself. Trivially, then the “world” is what should be understood intuitively as the maximum individual as part of itself. There is nothing more than a maximum individual, because two individuals are identical if and only if they have exactly the same parts.

2. The specification of the maximum individual is exhaustive, in the sense that, given a set of all universals $\{P_1, P_2, \dots, P_n\}$ that are attributed to each party, either the universal P_i will be attributed or not to that universal P_i (for a universal $P_i \in \{P_1, P_2, \dots, P_n\}$, of course). Then, the form by which a universal structural becomes a maximum, is because it will specify each determination held by each part of a possible world, through clauses in this way:

Definition 12. *A possible world is given by an exhaustive description of all its parts. In turn, the parties from which a possible world is made can be fully encoded with the full specification of each of its parts, then with the full specification of each of the parts of those parties, etc.*

It is perfectly acceptable in the series of universals P_1, P_2, \dots, P_n that appear in 2, that other clauses that have the same form of 2 are found. As can be seen, it could happen that the coding of a possible world (e.g., the actual world) is performed by an infinitely complex universal. For example, if the parties can only be coded with the specification of parts and these in turn by specifying their respective parties, and then these others should refer to their parts, and the parts of the parties, and the parties of the parts of the parties, and so on.

It may be that the most universal structure is so complex that is not expressible by any formula in a natural or artificial language, even if it is infinite. This has no importance. A universal is a type of entity that may or may not be known by us and may or may not be cognizable by us, without limiting its existence somehow. Well, how can they affect Cantor paradoxes to the modal theory based on universals? First, this theory is affected by the arguments, of the form of C5 (i.e., the arguments rejecting the existence of a set of all essences). In the modal theory based on universals, “possible objects” should not appear. Naturally there are objects existing in the actual world, but nothing more.

The possible worlds are not simple instantiated universals (with one exception). Consider for example a universal that if instantiated would appear that Pegasus would be winged, Pegasus then should appear represented by his individual essence and not his own person. Similarly, then, in the theory of Plantinga using this modal theory, individual essences must be used as the domain on which modal statements are quantified.

As already indicated above, the general notion that a state of affairs like the winged Pegasus can be included as an integral clause of a maximum structural universal, as there is a universal (structural, naturally) that is to be the horse Pegasus, and there will be another universal structure consisting of winged creatures, so it can define the universal of being something (Pegasus and winged). Using a set of individual sentences, this generates a Cantor paradox of type C5.

A second Cantor problem that is much more obvious to the modal concept based on universals is the type C7, and it should be applied to the universal set of all universals and to complex constructions from other simpler universals. It therefore seems that a complex universal or universal structure can be generated for each subset of any universal.

5. Conclusions

As we can observe, the Cantor paradoxes seem to affect actualistic modal conceptions in various ways, as well as certain ideas that are deeply rooted in the metaphysical tradition. In fact, it seems that it was inconsistent to think of the world as the set of all states of affairs or facts, and also think about a set of all propositions made true by virtue of this totality of states of affairs, and to think that these totalities of facts and true propositions could be known by an entity that knows “*everything*”. It seems that the problem does not affect actualistic modal concepts because they are modal theories, but the problem seems to stem from the representation of facts by means of all *possible worlds*. Similar issues raise these Cantor paradoxes because logicians traditionally reject unrestricted quantification ranging over all existing entities.

A characteristic principle of set theory is the absence of a universal set consisting of all sets. Likewise, it seems that these same principles that drove the development of iterative conceptions of sets lead to a fundamental constraint on our ontologies. That is, it appears we must accept that there is not clarity concerning *the whole*, and *there is no clarity about the universe*. Of course, this does not mean that we ordinarily use universal quantifications in our ordinary speech, but we must always assume that these universal quantifications are tacitly restricted to a specific domain of entities. The problem that arises here is not that the expression “*all*” cannot be used in these ordinary senses which are implicitly restricted to a certain domain. The problem arises for the unrestricted use of “*all*”. So, if it is not consistent to mention all the current states of things, or to talk of all propositions that are actually true, it is not surprising that it does not seem consistent to mention all the possible states of affairs or all propositions belonging to a set of maximally consistent propositions.

Thus, the Cantor paradoxes seem to require a uniform and general treatment. It has been common in the literature devoted specifically to issues of modal metaphysics to propose special solutions for one or another type of Cantor paradox. This approach is apparently wrong: there are no actualist theories immune to problems of cardinality; there is also no ontological conception of the “world” that is free of these difficulties. The solution—or solutions—should be applicable to all these conceptions alike. Indeed, all modal concepts under consideration explain the modal facts appealing to a totality of possible worlds, which are “*alternative forms which could constitute the world*” but differ in how nature is conceived by these “*alternative forms*”. In any case, possible worlds are always specifications for how the world could be. Then, if there are possible worlds, there is a current world (or in the case of Lewis, there are simply “worlds” in the plural). If arguments C1, C2, and C6 question the existence of a world as a set of all truths, the set of all states of affairs or the set of all existing institutions, then these Cantor arguments will be a reason to deny that there is a plurality of possible worlds. As indicated above, this is not because there is any particular difficulty in the way such possible worlds are designed, but simply because they implicitly use a conceptually incoherent category.

Tacitly, actualistic conceptions assume there is a whole world, which may be different. Cantor arguments would show that there is no such entity (i.e., *the world*) and it cannot exist, so there are no alternative forms which could be that entity. The problem therefore has an absolutely general nature and is not restricted to modal metaphysics.

Cantor paradoxes do not offer particular advantages to some or any of the modal theories. The problems derived from Cantor paradoxes, although they have focused considerable attention to difficulties in the principle of actualistically different theories, are usually against a particular theory or another, and are not matters of special urgency. The underlying problem is quite general and affects anyone who intends to use the notion of “world” in their ontology. Even a philosopher who rejects all metaphysical forms will have to address this issue. It is therefore not a specific difficulty of modal concepts, and it is also not a difficulty to allow or grant advantages or signify some disadvantages for some modal theory. In short, modal metaphysicians may well bracket this issue to concentrate their energies on issues that have real relevance for its development.

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